Finsler Metrics with Bounded Cartan Torsions

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Abstract

The norm of Cartan torsion plays an important role for studying of immersion theory in Finsler geometry. Indeed, Finsler manifold with unbounded Cartan torsion can not be isometrically imbedded into any Minkowski space. In this paper, we find two subclasses of (α, β)-metrics which have bounded Cartan torsion. Then, we give two subclasses of (α, β)-metrics whose bound on the Cartan torsions are independent of the norm of β.

Keywords: Cartan Torsion, (α, β)-metric, Randers metric.[1]

1 Introduction

One of fundamental problems in Finsler geometry is wether or not every Finsler manifold can be isometrically immersed into a Minkowski space. The answer is affirmative for Riemannian manifolds [15]. For Finsler manifolds, the problem under some conditions was considered by Burago-Ivanov, Gu and Ingarden [4] [8][9][10]. In [18], Shen proved that Finsler manifold with unbounded Cartan torsion can not be isometrically imbedded into any Minkowski space. Thus the norm of Cartan torsion plays a important role for studying of immersion theory in Finsler geometry. For a Finsler manifold (M, F), the second and third order derivatives of $\frac{1}{2}F_x^2$ at $y \in T_xM_0$ are fundamental form $\mathbf{g}_y$ and the Cartan torsion $\mathbf{C}_y$ on T_xM, respectively [1]. The Cartan torsion was first introduced by Finsler [7] and emphased by Cartan [5]. For the Finsler metric F, one can defines the norm of the Cartan torsion $\mathbf{C}$ as follows

$$||\mathbf{C}|| = \sup_{F(y)=1, v\neq 0} \frac{|\mathbf{C}_y(v, v, v)|}{[\mathbf{g}_y(v, v)]^{\frac{3}{2}}}.$$

The bound for two dimensional Randers metrics $F = \alpha+\beta$ is verified by Lackey [2]. Then, Shen proved that the Cartan torsion of Randers metrics on a manifold M of dimension $n \geq 3$ is uniformly bounded by $3/\sqrt{2}$ [17]. In [14], Mo-Zhou extend his result to a general Finsler metrics, $F = \frac{(\alpha+\beta)^m}{\alpha^{m-1}}$ $(m \in [1, 2])$. Recently, the first two authors find a relation between the norm of Cartan and mean Cartan torsions of Finsler metrics defined by a Riemannian metric and a 1-form on a manifold [21]. They prove that generalized Kropina metrics $F = \frac{\alpha^{m+1}}{\beta^m}$, $(m \neq 0)$ have bounded Cartan torsion. It turns out that every C-reducible Finsler metric has bounded Cartan torsion.

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All of above metrics are special Finsler metrics so- called (α, β)-metrics. An (α, β)-metric is a Finsler metric on M defined by $F := \alpha\phi(s)$, where $s = \beta/\alpha$, $\phi = \phi(s)$ is a C^∞ function on the $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M.

In this paper, we consider a special (α, β)-metric, called the generalized Randers metric $F = \sqrt{c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2}$ on a manifold M. By putting $c_1 = c_2 = c_3 = 1$, we get the Randers metric. First, we prove the following.

Theorem 1.1. *Let $F = \sqrt{c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2}$ be the generalized Randers metric on a manifold M, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric, $\beta = b_i(x)y^i$ is an 1-form on M and c_1, c_2, c_3 are real constants such that $c_2^2 < c_1c_3$ and $c_1^2 > |c_2(3c_1 + c_3)|$. Then F has bounded Cartan torsion.*

One of important (α, β)-metrics is Berwald metric which was introduced by L. Berwald on unit ball $U = B^n$ [3]. Berwald's metric can be expressed in the form $F = \lambda(\alpha + \beta)^2/\alpha$, where

$$\alpha = \frac{\sqrt{(1 - |x|^2)|y|^2 - < x, y >^2}}{1 - |x|^2}, \ \ \beta = \frac{< x, y >}{1 - |x|^2}, \ \ \lambda = \frac{1}{1 - |x|^2}, \ \ y \in T_xB^n \simeq \mathbb{R}^n$$

and $<, >$ and $|.|$ denote the Euclidean inner product and norm on $\mathbb{R}^n$, respectively. The Berwalds metric has been generalized by Shen to an arbitrary convex domain $U \subset R^n$ [20]. As an extension of the Berwald metric, we consider the metric $F = c_1\alpha + c_2\beta + c_3\beta^2/\alpha$ where $c_1, c_2, c_3 \in \mathbb{R}$. Then we prove the following.

Theorem 1.2. *Let $F = c_1\alpha + c_2\beta + c_3\beta^2/\alpha$ be an (α, β)-metric on a manifold M, where $\alpha := \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric, $\beta := b_i(x)y^i$ is an 1-form on M and c_1, c_2, c_3 are real constants such that $c_2^2 < 4c_1c_3$ and $|c_1| > |c_3|$. Then F has bounded Cartan torsion.*

For a vector $y \in T_xM_0$, the Riemann curvature $R_y : T_xM \to T_xM$ is defined by $R_y(u) := R^i{}_k(y)u^k \frac{\partial}{\partial x^i}$, where

$$R^i{}_k(y) = 2G^i_{x^k} - G^i_{x^j y^k} y^j + 2G^j G^i_{y^j y^k} - G^i_{y^j} G^j_{y^k}.$$

where $G^i = \frac{1}{4} g^{il}[(F^2)_{x^k y^l} y^k - (F^2)_{x^l}]$ are called the spray coefficients. The family $R := \{R_y\}_{y \in TM_0}$ is called the Riemann curvature. There are many Finsler metrics whose Riemann curvature in every direction is quadratic. A Finsler metric F is said to be Berwald metric and R-quadratic metric if G^i and R_y is quadratic in $y \in T_xM$ at each point $x \in M$, respectively. In [17], Shen proved that every complete R-quadratic manifold with bounded Cartan torsion is Landsbergian. He proved that a regular (α, β)-metric is Landsbergian if and only if it is Berwaldian [19]. Thus, we can conclude the following.

Corollary 1.1. *Let $F_1 = c_1\alpha + c_2\beta + c_3\beta^2/\alpha$, $(c_2^2 < 4c_1c_3, |c_1| > |c_3|)$ and $F_2 = \sqrt{c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2}$, $(c_2^2 < c_1c_3,\ c_1^2 > |c_2(3c_1 + c_3)|)$ are R-quadratic Finsler metrics on a complete manifold M. Then F_1 and F_2 are Berwaldian.*

By Theorems 1.1, it follows that if $c_2^2 = c_1c_3$ then the norm of Cartan torsion of Finsler metric $F = \sqrt{c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2}$ is independent of $b := ||\beta||_\alpha = \sqrt{b_i b^i}$, where $b^i = a^{ji}b_j$. It is an interesting problem, to find a subclass of (α, β)-metrics whose bound on the Cartan torsion is independent of b. In the final section, we give two subclasses of (α, β)-metrics whose bound on the Cartan torsions are independent of b.

2 Preliminaries

Let M be an n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, by $TM = \cup_{x\in M} T_xM$ the tangent bundle of M, and by $TM_0 = TM \setminus \{0\}$ the slit tangent bundle of M. A Finsler metric on M is a function $F : TM \to [0, \infty)$ which has the following properties:
(i) F is C^∞ on TM_0;
(ii) F is positively 1-homogeneous on the fibers of the tangent bundle of M;
(iii) for each $y \in T_xM$, the following quadratic form $\mathbf{g}_y$ on T_xM is positive definite,

$$\mathbf{g}_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s\partial t}\left[F^2(y + su + tv)\right]|_{s,t=0}, \quad u, v \in T_xM.$$

Let $x \in M$ and $F_x := F|_{T_xM}$. We define $\mathbf{C}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\left[\mathbf{g}_{y+tw}(u, v)\right]|_{t=0}, \quad u, v, w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM_0}$ is called the Cartan torsion. It is well known that $\mathbf{C} = 0$ if and only if F is Riemannian.

For $y \in T_xM_0$, define mean Cartan torsion $\mathbf{I}_y$ by $\mathbf{I}_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$. By Deicke's theorem, F is Riemannian if and only if $\mathbf{I}_y = 0$ [6].

Let (M, F) be a Finsler manifold. For $y \in T_xM_0$, define the Matsumoto torsion $\mathbf{M}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by $\mathbf{M}_y(u, v, w) := M_{ijk}(y)u^iv^jw^k$, where

$$M_{ijk} := C_{ijk} - \frac{1}{n+1}\{I_ih_{jk} + I_jh_{ik} + I_kh_{ij}\},$$

and $h_{ij} := g_{ij} - \frac{1}{F^2}g_{ip}y^pg_{jq}y^q$ is the angular metric. A Finsler metric F is said to be C-reducible, if $\mathbf{M}_y = 0$ [12]. Matsumoto proves that every Randers metric satisfies $\mathbf{M}_y = 0$. Later on, Matsumoto-Hōjō prove that the converse is true too.

Lemma 2.1. ([13]) A Finsler metric F on a manifold of dimension $n \geq 3$ is a Randers metric if and only if the Matsumoto torsion vanish.

Let $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ be a Riemannian metric, and $\beta = b_i(x)y^i$ be a 1-form on M with $\|\beta\| = \sqrt{a^{ij}b_ib_j} < 1$. The Finsler metric $F = \alpha + \beta$ is called a Randers metric, which has important applications both in mathematics and physics.

For a Finsler metric $F = F(x, y)$ on a smooth manifold M, geodesic curves are characterized by the system of second order differential equations

$$\frac{d^2x^i}{dt^2} + 2G^i\big(x, \frac{dx}{dt}\big) = 0,$$

where the local functions $G^i = G^i(x, y)$ are called the spray coefficients, and given by

$$G^i(x, y) := \frac{1}{4}g^{il}\Big[\frac{\partial^2F^2}{\partial x^k\partial y^l}y^k - \frac{\partial F^2}{\partial x^l}\Big].$$

In a standard local coordinates (x^i, y^i) in TM, the vector field $\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i}$ is called the spray of F. A Finsler metric F is called a Berwald metric, if G^i are quadratic in $y \in T_xM$ for any $x \in M$. The Berwald spaces can be viewed as Finsler spaces modeled on a single Minkowski space [16].

3 Proof of Theorem 1.1

Let us first consider the case of $dimM = 2$. There exists a local ortonormal coframe $\{\omega_1, \omega_1\}$ of Riemannian metric α. So α^2 can be wrote as

$$\alpha^2 = \omega_1^2 + \omega_2^2.$$

If we denote $\alpha = \sqrt{a_{ij} y^i y^j}$ where $y = \sum_{i=1}^{2} y^i e_i$ and $\{e_i\}$ is the dual frame of $\{\omega_i\}$ then $a_{ij} = \delta_{ij}$ and $a^{ij} = \delta^{ij}$. Adjust coframe $\{\omega_1, \omega_1\}$ properly such that

$$\beta = k\omega_1.$$

Then $b_1 = k$ and $b_2 = 0$ where $\beta = \sum_{i=1}^{2} b_i y^i$. Hence

$$\|\beta\|_\alpha := \sqrt{a^{ij} b_i b_j} = k.$$

For an arbitrary tangent vector $y = ue_1 + ve_2 \in T_pM$, we can obtain that

$$\alpha(p, y) = \sqrt{u^2 + v^2}, \qquad \beta(p, y) = ku,$$
$$F(p, y) = \sqrt{c_1(u^2 + v^2) + 2c_2 ku\sqrt{u^2 + v^2} + c_3 k^2 u^2}.$$

Assume that $y^\perp$ satisfies:

$$\mathbf{g}_y(y, y^\perp) = 0, \quad \mathbf{g}_y(y^\perp, y^\perp) = F^2(p, y). \tag{1}$$

Obviously $y^\perp$ is unique because the metric is non-degenerate. The frame $\{y, y^\perp\}$ is called the Berwald frame.

Let

$$y = r\cos(\theta)e_1 + r\sin(\theta)e_2$$

i.e.

$$u = r\cos(\theta), \qquad v = r\sin(\theta).$$

Plugging the above expression into (1) and computing by Maple program (see Section 5.1.1) yields

$$y^\perp = \frac{r\big(-\sin(\theta)(c_2 k\cos(\theta) + c_1), c_3 k^2 \cos(\theta) + c_2 k + c_1\cos(\theta) + c_2 k\cos(\theta)^2\big)}{\sqrt{c_1 c_3 k^2 + c_2 c_3 k^3 \cos(\theta)^3 + 3c_2^2 k^2 \cos(\theta)^2 + 3c_1 c_2 k\cos(\theta) - c_2^2 k^2 + c_1^2}}. \tag{2}$$

By the definition of the bound of Cartan torsion, it is easy to show that for the Berwald frame $\{y, y^\perp\}$,

$$\|\mathbf{C}\,\|_p = \sup_{y \in T_pM_0} \xi(p, y),$$

where

$$\xi(p, y) := \frac{F(p, y)|\mathbf{C}_y(y^\perp, y^\perp, y^\perp)|}{|\mathbf{g}_y(y^\perp, y^\perp)|^{\frac{3}{2}}}.$$

Again computing by Maple program (see Subsection 5.2.1 below), we obtain

$$\xi(p,y)=\frac{3}{2}\Big|\frac{c_2k\sin(\theta)\big(c_1+2c_2k\cos(\theta)+c_3k^2\cos(\theta)^2\big)^2}{(c_1c_3k^2+c_2c_3k^3\cos(\theta)^3+3c_2^2k^2\cos(\theta)^2+3c_1c_2k\cos(\theta)-c_2^2k^2+c_1^2)^{\frac{3}{2}}}\Big|$$

Define two functions on $[0,1]\times[-1,1]$ by following

$$f(k,x):=c_1c_3k^2+c_2c_3k^3x^3+3c_2^2k^2x^2+3c_1c_2kx-c_2^2k^2+c_1^2,$$
$$g(k,x):=\frac{3}{2}\frac{c_2k\sqrt{1-x^2}\big(c_1+2c_2kx+c_3k^2x^2\big)^2}{f(k,x)^{\frac{3}{2}}}.$$

Hence

$$\|\mathbf{C}\,\|_p=\max_{0\leq\theta\leq 2\pi}|g(k,\cos\theta)|. \tag{3}$$

For a fixed $k=k_0$ ($k_0\in[0,1]$), we have

$$\frac{\partial}{\partial x}f(k_0,x)=3c_1c_2k_0+6c_2^2k_0^2x+3c_2c_3k_0^3x^2.$$

So from $\frac{\partial}{\partial x}f(k_0,x)=0$, we have

$$x\in\Big\{\frac{-c_2+\sqrt{c_2^2-c_1c_3}}{c_3k_0},\frac{-c_2-\sqrt{c_2^2-c_1c_3}}{c_3k_0}\Big\}.$$

Because of $c_2^2<c_1c_3$, we conclude that $f(k_0,x)$ is ascending or descending. So for $x\in[-1,1]$ we have

$$f(k_0,x)\geq\min\{f(k_0,-1),f(k_0,1)\}.$$

By a simple computation, we have

$$f(k_0,1)=c_1^2+3c_1c_2k_0+c_1c_3k_0^2+2c_2^2k_0^2+c_2c_3k_0^3,$$
$$f(k_0,-1)=c_1^2-3c_1c_2k_0+c_1c_3k_0^2+2c_2^2k_0^2-c_2c_3k_0^3.$$

Since $c_1c_3\geq 0$ and $c_1^2>|c_2(3c_1+c_3)|$, then we have

$$f(k_0,1)>0,\quad f(k_0,-1)>0.$$

So for $k\in[0,1]$ and $x\in[-1,1]$, we have $f(k,x)>0$. Thus $g(k,x)$ is continuous in $[0,1]\times[-1,1]$ and has a upper bound.

In higher dimensions, the definition of the Cartan torsion's bound at $p\in M$ is

$$\|\mathbf{C}\,\|_p=\sup_{y,u\in T_pM}\frac{F(p,y)|\mathbf{C}_y(u,u,u)|}{|\mathbf{g}_y(u,u)|^{\frac{3}{2}}}.$$

Considering the plane $P=span\{u,y\}$, from the above conclusion we obtain $\|\mathbf{C}\,\|_p$ is bounded. Furthermore, the bound is independent of the plane $P\subset T_pM$ and the point $p\in M$. Hence the Cartan torsion is also bounded. This completes the proof.

4 Proof of Theorem 1.2

In this section, we are going to prove the Theorem 1.2. Let us first consider the case of $dimM = 2$. By the similar method used in proof of Theorem 1.1, for an arbitrary tangent vector $y = ue_1 + ve_2 \in T_pM$ we can obtain that

$$\alpha(p, y) = \sqrt{u^2 + v^2}, \qquad \beta(p, y) = ku$$
$$F(p, y) = c_1\sqrt{u^2 + v^2} + c_2ku + c_3\frac{k^2u^2}{\sqrt{u^2 + v^2}}.$$

Using the Maple program (see Section 5.1.2), we get

$$y^{\perp} = \frac{r\big(- \sin(\theta)(c_1 - c_3k^2\cos(\theta)^2), c_1\cos(\theta) + kc_2 + 2c_3k^2\cos(\theta) - c_3k^2\cos(\theta)^3\big)}{\sqrt{(-3k^2c_3\cos(\theta)^2 + 2k^2c_3 + c_1)(k^2c_3\cos(\theta)^2 + kc_2\cos(\theta) + c1)}} \tag{4}$$

Again computing by Maple program (see subsection 5.2.2), we obtain

$$\xi(p, y) = \frac{3}{2}\Big|\frac{k\sin(\theta)\big(- c_1c_2 - 4k^3c_3^2\cos(\theta) + 8c_3^2k^3\cos(\theta)^3 - 2k^2c_2c_3 + 5c_2c_3k^2\cos(\theta)^2\big)}{(3c_3k^2\cos(\theta)^2 - 2c_3k^2 - c1)\sqrt{(-3c_3k^2\cos(\theta)^2 + 2c_3k^2 + c_1)(c_1 + c_2k\cos(\theta) + c_3k^2\cos(\theta)^2)}}\Big|.$$

Define three functions on $[0, 1] \times [-1, 1]$

$$f_1(k, x) := 3c_3k^2x^2 - 2c_3k^2 - c_1,$$
$$f_2(k, x) := c_1 + c_2kx + c_3k^2x^2,$$
$$g(k, x) := \frac{3}{2}\frac{k\sqrt{1 - x^2}\big(- c_1c_2 - 4k^3c_3^2x + 8c_3^2k^3x^3 - 2k^2c_2c_3 + 5c_2c_3k^2x^2\big)}{f_1(k, x)\sqrt{-f_1(k, x)f_2(k, x)}}.$$

Hence

$$\|\mathbf{C}\|_p = \max_{0\leq\theta\leq 2\pi} |g(k, \cos\theta)|. \tag{5}$$

For a fixed $k = k_0$ ($k_0 \in [0, 1]$) the roots of $f_2(k_0, x)$ are

$$\Big\{\frac{1}{2}\frac{-c_2 + \sqrt{c_2^2 - 4c_1c_3}}{c_3k_0}, \frac{1}{2}\frac{-c_2 - \sqrt{c_2^2 - 4c_1c_3}}{c_3k_0}\Big\}.$$

Because of $c_2^2 < 4c_1c_3$, for $x \in [-1, 1]$ and $k \in [0, 1]$ we have $f_2(k_0, x) \neq 0$. If $c_3 \geq 0$ because of $c_2^2 < 4c_1c_3$ we get $c_1 \geq 0$ and the maximum of $f_1(k, x)$ in $[-1, 1]$ occurred in $x \in \{-1, 1\}$. By simple computation we have:

$$f_1(k_0, -1) = f_1(k_0, 1) = c_3k_0^2 - c_1. \tag{6}$$

By the assumption, we have $|c_1| > |c_3|$ so we conclude that

$$f_1(k_0, -1) = f_1(k_0, 1) < 0.$$

If $c_3 \leq 0$ because of $c_2^2 < 4c_1c_3$ we have $c_1 \leq 0$ and the minimum of $f_1(k, x)$ in $[-1, 1]$ occurred in $x \in \{-1, 1\}$. By the assumption, we have $|c_1| > |c_3|$ so by (6) we conclude that

$$f_1(k_0, -1) = f_1(k_0, 1) > 0.$$

So for $k \in [0, 1]$ and $x \in [-1, 1]$, we have

$$f_1(k, x) \neq 0.$$

Then $g(k, x)$ is continuous in $[0, 1] \times [-1, 1]$ and has a upper bound. For the higher dimensions, proof is similar to the 2-dimensional case.

5 Maple Programs

In this section, we are give the Maple programs which used to proving the Theorems 1.1 and Theorem 1.2.

5.1 Berwald Frame

The special and useful Berwald frame was introduced and developed by Berwald. Let (M, F) be a two-dimensional Finsler manifold. We study two dimensional Finsler space and define a local field of orthonormal frame (ℓ^i, m^i) called the Berwald frame, where $\ell^i = y^i/F(y)$, m^i is the unit vector with $\ell_i m^i = 0$, $\ell_i = g_{ij}\ell^i$ and g_{ij} is defined by $g_{ij} = \ell_i\ell_j + m_i m_j$.

5.1.1 Berwald Frame of $F = \sqrt{c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2}$

$> restart;$
$> with(linalg):$
$> F := sqrt\big((c[1]) * (u\hat{}2 + v\hat{}2) + 2 * (c[2]) * k * u * sqrt(u\hat{}2 + v\hat{}2) + (c[3]) * k\hat{}2 * u\hat{}2\big):$
$> g := simplify(1/2 * hessian(F\hat{}2, [u, v])):$
$> gr := simplify(subs(u = cos(theta), v = sin(theta), g)):$
$> y := vector(2, [r * cos(theta), r * sin(theta)]);$

$$y := [u = r\cos(\theta), v = r\sin(\theta)]$$

$> yp := vector(2):$
$> eq := simplify(evalm(transpose(y)\& * gr\& * yp)) = 0:$
$> x := solve(eq, yp[1]);$

$$x = -\frac{yp_2 sin(\theta)(c_2 k\cos(\theta) + c_1)}{c_3 k^2\cos(\theta) + c_2 k + c_1\cos(\theta) + c_2 k cos(\theta)^2}$$

$> ny := simplify(r\hat{}2 * subs(u = cos(theta), v = sin(theta), F\hat{}2));$

$$ny := r^2(c_1 + 2c_2 k\cos(\theta) + c_3 k^2\cos(\theta)^2)$$

$> yp[1] := -\sin(theta) * (c[2] * k * \cos(theta) + c[1]):$
$> yp[2] := c[3]*k\hat{}2*\cos(theta)+c[2]*k+c[1]*\cos(theta)+c[2]*k*\cos(theta)\hat{}2:$
$> nyp := simplify(evalm(transpose(yp)\& * gr\& * yp)):$
$> lambda := simplify(sqrt(r\hat{}2 * nyp/ny)/r):$
$> yp[1] := yp[1]/lambda:$
$> yp[2] := yp[2]/lambda:$
$> print(yp);$

$$\Big[-\frac{\sin(\theta)(c_2 k\cos(\theta) + c_1)r}{\sqrt{c_1c_3k^2 + c_2c_3k^3\cos(\theta)^3 + 3c_2^2k^2\cos(\theta)^2 + 3c_1c_2k\cos(\theta) - c_2^2k^2 + c_1^2}},$$
$$\frac{(c_3k^2\cos(\theta) + c_2k + c_1\cos(\theta) + c_2k\cos(\theta)^2)r}{\sqrt{c_1c_3k^2 + c_2c_3k^3\cos(\theta)^3 + 3c_2^2k^2\cos(\theta)^2 + 3c_1c_2k\cos(\theta) - c_2^2k^2 + c_1^2}}\Big]$$

5.1.2 Berwald Frame of $F = c_1\alpha + c_2\beta + c_3\beta^2/\alpha$

$> restart;$
$> with(linalg):$
$> F := c[1]*sqrt(u\hat{}2+v\hat{}2)+c[2]*k*u+(c[3]*k\hat{}2*u\hat{}2)/(sqrt(u\hat{}2+v\hat{}2)):$
$> g := simplify(1/2*hessian(F\hat{}2,[u,v])):$
$> gr := simplify(subs(u = cos(theta), v = sin(theta), g)):$
$> y := vector(2,[r*cos(theta), r*sin(theta)]);$

$$y := [u = r\cos(\theta), v = r\sin(\theta)]$$

$> yp := vector(2):$
$> eq := simplify(evalm(transpose(y)\&*gr\&*yp)) = 0:$
$> x := solve(eq, yp[1]);$

$$x = -\frac{(c_3k^2\cos(\theta)^2 - c_1)yp_2\sin(\theta)}{c_3k^2\cos(\theta)^3 - 2c_3k^2\cos(\theta) - c_1\cos(\theta) - c_2k}$$

$> ny := simplify(r\hat{}2*subs(u = cos(theta), v = sin(theta), F\hat{}2));$

$$ny := r^2(c_1 + c_2k\cos(\theta) + c_3k^2\cos(\theta)^2)^2$$

$> yp[1] := -(-c[1] + c[3]*k\hat{}2*\cos(\theta)\hat{}2)*\sin(\theta):$
$> yp[2] :=: -c[1]*\cos(theta) - k*c[2] - 2*c[3]*k\hat{}2*\cos(theta) + c[3]*k\hat{}2*$ $\cos(theta)\hat{}3:$
$> nyp := simplify(evalm(transpose(yp)\&*gr\&*yp)):$
$> lambda := simplify(sqrt(r\hat{}2*nyp/ny)/r):$
$> yp[1] := yp[1]/lambda:$
$> yp[2] := yp[2]/lambda:$
$> print(yp);$

$$\Big[\frac{-(-c_1 + c_3k^2\cos(\theta))r\sin(\theta)}{\sqrt{(-3k^2c_3\cos(\theta)^2 + 2k^2c_3 + c_1)(k^2c_3\cos(\theta)^2 + kc_2\cos(\theta) + c1)}}$$
$$, \frac{(-c_1\cos(\theta) - c_2k - 2c_3k^2\cos(\theta) + c_3k^2\cos(\theta)^3)r}{\sqrt{(-3k^2c_3\cos(\theta)^2 + 2k^2c_3 + c_1)(k^2c_3\cos(\theta)^2 + kc_2\cos(\theta) + c1)}}\Big]$$

5.1.3 The Method of Computation

Step 1: Solve the equation $\mathbf{g}_y(y, y^\perp) = 0$.

$$(x, yp_{[2]}) = (\frac{yp_{[2]}yp_{[1]}}{yp_{[2]}}, yp_{[2]})$$

and $yp := (yp_{[1]}, yp_{[2]})$ is a particular solution.

Step 2: Assume that $y^\perp = \frac{1}{\lambda}yp$ is the satisfied solution. Notice that

$$\mathbf{g}_y(y^\perp, y^\perp) = F^2(y) := ny$$

Then we get

$$\lambda = \sqrt{\frac{nyp}{ny}}$$

which nyp is defined by

$$nyp := \mathbf{g}_y(yp, yp)$$

Step 3: Plug these results into $y^\perp$, we get the Berwald frame $\{y, y^\perp\}$.

5.2 Computation of $\xi(p, y)$

Here, we are going to compute $\xi(p, y)$ for the Finsler metrics defined by $F = \sqrt{c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2}$ and $F = c_1\alpha + c_2\beta + c_3\beta^2/\alpha$, where c_1, c_2 and c_3 are real numbers.

5.2.1 Computation of $\xi(p, y)$ for $F = \sqrt{c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2}$

$> nyp := simplify(evalm(transpose(yp)\& * gr\& * yp));$

$$nyp := r^2\left(c_1 + 2c_2k\cos(\theta) + c_3k^2\cos(\theta)^3\right)$$

$> bc := factor(abs(simplify(r\hat{}2 * subs(t = 0, q = 0, p = 0, diff(subs(u =$
$\cos(theta) + t * yp[1]/$
$> r + q * yp[1]/r + p * yp[1]/r,$
$> v = \sin(theta) + t * yp[2]/r + q * yp[2]/r + p * yp[2]/r, F\hat{}2/4), [t, q, p])))/nyp));$

$$bc := \frac{3}{2}\left|\frac{c_2k\sin(\theta)\left(c_1 + 2c_2k\cos(\theta) + c_3k^2\cos(\theta)^2\right)^2}{(c_1c_3k^2 + c_2c_3k^3\cos(\theta)^3 + 3c_2^2k^2\cos(\theta)^2 + 3c_1c_2k\cos(\theta) - c_2^2k^2 + c_1^2)^{\frac{3}{2}}}\right|$$

5.2.2 Computation of $\xi(p, y)$ for $F = c_1\alpha + c_2\beta + c_3\beta^2/\alpha$

$> nyp := simplify(evalm(transpose(yp)\& * gr\& * yp));$

$$nyp := r^2(c_1^2 + 2c_1c_2k\cos(\theta) + 2c_1c_3k^2\cos(\theta)^2 + c_2^2k^2\cos(\theta)^2 + 2c_2c_3k^3\cos(\theta)^3 + c_3^2k^4\cos(\theta)^4)$$

$> bc := factor(abs(simplify(r\hat{}2 * subs(t = 0, q = 0, p = 0, diff(subs(u =$
$\cos(theta) + t * yp[1]/$
$> r + q * yp[1]/r + p * yp[1]/r,$
$> v = \sin(theta) + t * yp[2]/r + q * yp[2]/r + p * yp[2]/r, F\hat{}2/4), [t, q, p])))/nyp));$

$$bc := \frac{3}{2}\left|\frac{k\sin(\theta)\left(-c_1c_2 - 4k^3c_3^2\cos(\theta) + 8c_3^2k^3\cos(\theta)^3 - 2k^2c_2c_3 + 5c_2c_3k^2\cos(\theta)^2\right)}{(3c_3k^2\cos(\theta)^2 - 2c_3k^2 - c1)\sqrt{(-3c_3k^2\cos(\theta)^2 + 2c_3k^2 + c_1)(c_1 + c_2k\cos(\theta) + c_3k^2\cos(\theta)^2)}}\right|$$

5.2.3 The method of computation:

Let

$$nyp := g_y(yp, yp) = g_y(y^\perp, y^\perp)$$

Then compute

$$bc = \frac{F(y)\mathbf{C}_y(y^\perp, y^\perp, y^\perp)}{g_y(y^\perp, y^\perp)^{\frac{3}{2}}}$$

This is prepared for estimating the bound of Cartan torsion.

6 Some Remarks

In this section, we will link our theorems to the results in [21] and discuss some related problems. In [21], the following is proved.

Theorem 6.1. ([21]) Let $F = \alpha\phi(s)$ be a non-Riemannian (α, β)-metric on a manifold M of dimension $n \geq 3$. Then the norm of Cartan and mean Cartan torsion of F satisfy in following relation

$$\|\mathbf{C}\| = \sqrt{\frac{3p^2 + 6p\ q + (n+1)q^2}{n+1}}\ \|\mathbf{I}\|, \tag{7}$$

where $p = p(x, y)$ and $q = q(x, y)$ are scalar function on TM satisfying $p+q = 1$ and given by following

$$p = \frac{n+1}{aA}\Big[s(\phi\phi'' + \phi'\phi') - \phi\phi'\Big], \tag{8}$$

$$a := \phi(\phi - s\phi'), \tag{9}$$

$$A = (n-2)\frac{s\phi''}{\phi - s\phi'} - (n+1)\frac{\phi'}{\phi} - \frac{(b^2 - s^2)\phi''' - 3s\phi''}{(b^2 - s^2)\phi'' + \phi - s\phi'}. \tag{10}$$

The Cartan tensor of an (α, β)-metric is given by following

$$C_{ijk} = \frac{p}{1+n}\{h_{ij}I_k + h_{jk}I_i + h_{ki}I_j\} + \frac{q}{\|\mathbf{I}\|^2}I_iI_jI_k. \tag{11}$$

where $p = p(x, y)$ and $q = q(x, y)$ are scalar functions on TM satisfying $p+q = 1$ and p is defined by (8). It is remarkable that, a Finsler metric is called semi-C-reducible if its Cartan tensor is given by the equation (11). It is proved that every non-Riemannian (α, β)-metric on a manifold M of dimension $n \geq 3$ is semi-C-reducible [11]. A Finsler metric F is said to be $C2$-like if $p = 0$ and is called C-reducible if $q = 0$.

It is well-known conclusive theorem that every C-reducible Finsler metric is Randers metric. Now, we have a natural problem: Are semi-C-reducible metrics necessarily (α, β)-metric?

Corollary 6.1. *Let $F = \alpha\phi(s)$ be a non-Randersian (α, β)-metric on a manifold M of dimension $n \geq 3$. Then F is not a C2-like metric.*

Proof. By Theorem 6.1, F is a C2-like metric if and only if ϕ satisfies the following

$$s(\phi\phi'' + \phi'\phi') - \phi\phi' = 0. \tag{12}$$

Solving (15), we obtain that

$$\phi = \sqrt{c_1s^2 + c_2},$$

where c_1 and c_2 are two real constant. For this ϕ, the (α, β)-metric $F = \alpha\phi(s)$ is Riemannian, which is a contradiction. ☐

Thus the converse of Theorem 6.1 is not true.

For a generalized Randers metric $F = \sqrt{c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2}$ on a manifold M, we have

$$\phi = \sqrt{c_1 + 2c_2 s + c_3 s^2}.$$

Then we get the following

$$a = c_1 + c_2 s$$
$$A = \frac{3(n-2)sk}{(c_1+c_2 s)\phi^4} - \frac{(n+1)(c_2+c_3 s)}{\phi^2} + \frac{3(c_c c_3 - c_2^2)s\phi^2 - 3(b^2-s^2)k}{\phi^2\big[(c_1+c_2 s)\phi^2 + (b^2-s^2)(c_c c_3 - c_2^2)s\big]},$$

where $k := (c_2^2 - c_1 c_3)(c_1 + c_3 s)$. Thus

$$p = -\frac{(n+1)c_2}{(c_1+c_2 s)A}. \tag{13}$$

Then we get the following.

Corollary 6.2. *Let $F = \sqrt{c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2}$ be a generalized Randers metric on a manifold M. Then the relation between the norm of Cartan and mean Cartan torsion of F satisfy in (7) where p is given by (13).*

For the metric $F = c_1\alpha + c_2\beta + c_3\beta^2/\alpha$, we have

$$\phi = c_1 + c_2 s + c_3 s^2.$$

Then we get the following

$$a = (c_1 - c_3 s^2)(c_1 + c_2 s + c_3 s^2)$$
$$A = \frac{2(n-2)c_3 s}{c_1 - c_3 s^2} - \frac{(n+1)(c_2 + 2c_3 s)}{c_1 + c_2 s + c_3 s^2} + \frac{6c_3 s}{(c_1 - c_3 s^2) + 2(b^2 - s^2)c_3}.$$

Thus

$$p = \frac{n+1}{(c_1 + c_3 s^2)(c_1 + c_2 s + c_3 s^2)A}\Big[3c_2 c_3 s^2 + 4c_3 s^3 - c_1 c_2\Big]. \tag{14}$$

By taking $c_3 = 0$, we have the Randers metric $F = c_1\alpha + c_2\beta$. In this case, by (14) we get $p = 1$ and $q = 0$. Thus for a Randers metric, we have the following

$$C_{ijk} = \frac{1}{1+n}\{h_{ij}I_k + h_{jk}I_i + h_{ki}I_j\}, \quad \text{and} \quad \|\mathbf{C}\| = \sqrt{\frac{3}{n+1}}\,\|\mathbf{I}\|.$$

Now, if we put $c_1 = c_3 = 1$ and $c_2 = 2$ then we get Berwald metric $F = \frac{(\alpha+\beta)^2}{\alpha}$. Similar to the Corollary 6.2, we get the following.

Corollary 6.3. *Let $F = c_1\alpha + c_2\beta + c_3\beta^2/\alpha$ be an (α,β)-metric on a manifold M. Then the relation between the norm of Cartan and mean Cartan torsion of F satisfy in (7) where p is given by (14).*

Now, let $c_2^2 = c_1 c_3$. Then

$$k = 0 \quad \text{and} \quad A = -\frac{(n+1)(c_2 + c_3 s)}{\phi^2}.$$

In this case, it is easy to see that the norm of Cartan torsions of Finsler metric $F=\sqrt{c_1\alpha^2+2c_2\alpha\beta+c_3\beta^2}$ is independent of b. It is an interesting problem, to find a subclass of (α,β)-metrics whose bound on the Cartan torsion is independent of $b=||\beta||_\alpha$. Here, we give some Finsler metrics with such property. For this work, we find all of solutions that for them the numerator of final sentence in (10) is vanishing, i.e.,

$$(b^2-s^2)\phi'''-3s\phi''=0. \tag{15}$$

The solutions of (15) are given by following

$$\phi_1=-\frac{d_1\sqrt{s^2-b^2}}{b^2}+d_2s+d_3 \tag{16}$$

and

$$\phi_2=\frac{d_1\sqrt{b^2-s^2}}{b^2}+d_2s+d_3, \tag{17}$$

where d_1,d_2,d_3 are constants. Then we get the following.

Theorem 6.2. *Let $F=\alpha\phi(s)$ are the (α,β)-metrics defined by (16) or (17). Then the norm of Cartan torsion of F is independent of $b=||\beta||$.*

The other simple answer to this question arise when numerator of final sentence in (10) is a multiplying factor of the denominator. In this case, there is a real constant λ such that the following holds

$$\frac{(b^2-s^2)\phi'''-3s\phi''}{(b^2-s^2)\phi''+\phi-s\phi'}=\lambda. \tag{18}$$

Thus we have the following ODE

$$\phi'''-\big(\lambda+\frac{3s}{b^2-s^2}\big)\phi''+\frac{s\ \lambda}{b^2-s^2}\phi'-\frac{\lambda}{b^2-s^2}\phi=0. \tag{19}$$

The solutions of (19) are given by following

$$\phi=c_1s+c_2\sqrt{b^2-s^2}+c_3\sqrt{b^2-s^2}\int\frac{e^{\lambda s}}{(b^2-s^2)^{\frac{3}{2}}}ds, \tag{20}$$

where c_1,c_2,c_3 are real constants. Therefore, we have the following.

Theorem 6.3. *Let $F=\alpha\phi(s)$ are the (α,β)-metrics defined by (20). Then the norm of Cartan torsion of F is independent of $b=||\beta||$.*

Open Problems. Some natural question arises as following:
(I) How large is the subclass of (α,β)-metrics which their norm of Cartan torsion are independent of $b=||\beta||$?
(II) The other question is to find all of (α,β)-metrics with bounded Cartan torsion.

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